stochastic processes and their applications

www.elsevier.com/locate/spa

Spectral decomposition for operator self-similar processes and their generalized domains of attraction

Mark M. Meerschaert[a,*], Hans-Peter Scheffler[b]

[a] *Department of Mathematics, University of Nevada, Reno, NV 89557-0045, USA*
[b] *Department of Mathematics, University of Dortmund, 44221 Dortmund, Germany*

Received 20 July 1998; received in revised form 5 May 1999; accepted 7 May 1999

Abstract

A stochastic process on a finite-dimensional real vector space is operator-self-similar if a linear time change produces a new process whose distributions scale back to those of the original process, where we allow scaling by a family of affine linear operators. We prove a spectral decomposition theorem for these processes, and for processes with these scaling limits. This decomposition reduces the study of these processes to the case where the growth behavior over time is essentially uniform in all radial directions. © 1999 Elsevier Science B.V. All rights reserved.

MSC: Primary 60F05; Secondary 60E07

Keywords: Operator-self-similar; Spectral decomposition; Regular variation

1. Introduction

An $\mathbb{R}^k$-valued stochastic process $\{X(t)\}_{t\geqslant 0}$ is called *operator-self-similar* if it is continuous in law and there exists a linear operator D on $\mathbb{R}^k$ and nonrandom vectors $\{d(t)\}_{t\geqslant 0}$ in $\mathbb{R}^k$ such that

$$\{X(ct)\} \stackrel{\mathrm{d}}{=} \{c^D X(t) + d(c)\}, \quad \forall c > 0, \tag{1}$$

where $\stackrel{\mathrm{d}}{=}$ means the equality of all finite-dimensional marginal distributions of the processes. The linear operator D is called an *exponent* of the operator-self-similar process, and $c^D = \exp(D \log c)$ where $\exp(A) = I + A + A^2/2! + \cdots$ is the usual exponential operator. The exponent D of an operator-self-similar process is not unique in general, because of possible symmetries. Theorem 2 of Hudson and Mason (1982) characterizes the set of exponents for a given operator-self-similar process. We always assume that the operator-self-similar process $\{X(t)\}$ is *proper*, meaning that the distribution of $X(t)$

* Corresponding author.
E-mail addresses: mcubed@unr.edu (M.M. Meerschaert), hps@mathematik.uni.dortmund.de (H.-P. Scheffler)

PII: S0304-4149(99)00048-4

is full for all $t > 0$. A probability distribution on $\mathbb{R}^k$ is *full* if it cannot be supported on any $d-1$-dimensional hyperplane.

We say that a stochastic process $\{Y(t)\}_{t\geqslant 0}$ belongs to the *generalized domain of attraction* of a proper process $\{X(t)\}_{t\geqslant 0}$ if there exist invertible linear operators $A(\lambda)$ on $\mathbb{R}^k$ and nonrandom vectors $a(\lambda) \in \mathbb{R}^k$ such that

$$\{A(\lambda)Y(\lambda t) + a(\lambda)\}_{t\geqslant 0} \overset{\mathrm{d}}{\Rightarrow} \{X(t)\}_{t\geqslant 0} \quad \text{as } \lambda \to \infty, \tag{2}$$

where $\overset{\mathrm{d}}{\Rightarrow}$ denotes the convergence of all finite-dimensional marginal distributions. In this case we write $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$. Hudson and Mason (1982) show that, for a proper stochastic process $\{X(t)\}_{t\geqslant 0}$ which is continuous in law, $\mathrm{GDOA}(\{X(t)\}) \neq \emptyset$ if and only if $\{X(t)\}$ is operator-self-similar. A proper operator-self-similar process always belongs to its own generalized domain of attraction.

The purpose of this note is to prove a *spectral decomposition theorem* for operator-self-similar processes and their generalized domains of attraction. It is clear from (1) that the exponent D governs the growth behavior over time of an operator-self-similar process. The spectral decomposition reduces the study of operator-self-similar processes and their generalized domains of attraction to the case where the growth behavior is essentially uniform in all radial directions. It can also be viewed as an extension of the corresponding result for operator stable laws and their generalized domains of attraction. In particular, if $\{X(t)\}$ is a process with stationary, independent, operator stable increments, then the spectral decomposition theorem of this paper can be derived as a consequence of Theorem 4.2 in Meerschaert (1991).

2. Spectral decomposition theorem

Suppose that $\{X(t)\}_{t\geqslant 0}$ is a proper operator-self-similar process on $\mathbb{R}^k$ and that D is an exponent of $\{X(t)\}$. Factor the minimal polynomial g of D into $g_1(x)\cdots g_p(x)$ where all roots of g_i have real part a_i and $a_i < a_j$ for $i < j$. Define $V_i = \mathrm{Ker}(g_i(D))$. Then $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^k$ into D-invariant subspaces, and we may write $D = D_1 \oplus \cdots \oplus D_p$ where $D_i : V_i \to V_i$ and every eigenvalue of D_i has real part equal to a_i. This spectral decomposition of the exponent D is a special case of the primary decomposition theorem of linear algebra, see for example [2]. Write $d(t) = d_1(t) + \cdots + d_p(t)$ and $X(t) = X_1(t) + \cdots + X_p(t)$ for $t \geqslant 0$ with respect to this direct sum decomposition. Using (1) along with the fact that the subspaces V_i are D-invariant, we have

$$\begin{aligned} X_1(ct) + \cdots + X_p(ct) &= X(ct) \\ &\overset{\mathrm{d}}{=} c^D X(t) + d(c) \\ &= (c^{D_1}X_1(t) + d_1(c)) + \cdots + (c^{D_p}X_p(t) + d_p(c)) \end{aligned}$$

for all $c > 0$. Projection onto V_i yields

$$\{X_i(ct)\}_{t\geqslant 0} \overset{\mathrm{d}}{=} \{c^{D_i}X_i(t) + d_i(c)\}_{t\geqslant 0}, \quad \forall c > 0 \tag{3}$$

for all $i = 1,2,\ldots,p$. Moreover, $\{X_i(t)\}$ is continuous in law and proper, so $\{X_i(t)\}$ is a proper operator-self-similar process on V_i with exponent D_i, where the real part of the eigenvalues of the exponent D_i are all equal to a_i. We say that $\{X_i(t)\}$ is a *spectrally simple* operator-self-similar process.

We say that a stochastic process $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$ is *spectrally compatible* with $\{X(t)\}$ if there exists a norming function $A(\lambda)$ such that (2) holds, and such that each of the subspaces V_i are $A(\lambda)$-invariant for all $\lambda > 0$. If this is the case, then it follows from (2) that

$$\{A_i(\lambda)Y_i(\lambda t) + a_i(\lambda)\}_{t\geqslant 0} \stackrel{\mathrm{d}}{\Rightarrow} \{X_i(t)\}_{t\geqslant 0} \quad \text{as } \lambda \to \infty, \tag{4}$$

so that $\{Y_i(t)\} \in \mathrm{GDOA}(\{X_i(t)\})$ for all $i = 1,\ldots,p$. This reduces the analysis of $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$ to the case of a spectrally simple limit.

Theorem 1. *Suppose that $\{X(t)\}_{t\geqslant 0}$ is a proper operator-self-similar process on $\mathbb{R}^k$ with exponent D, and that $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$. Then there exists an invertible linear operator T on $\mathbb{R}^k$ such that $\{TY(t)\}_{t\geqslant 0}$ is spectrally compatible with $\{X(t)\}$. Equivalently, $\{Y(t)\}$ is spectrally compatible with $\{T^{-1}X(t)\}_{t\geqslant 0}$ which is a proper operator-self-similar process with exponent $T^{-1}DT$.*

The proof of the spectral decomposition theorem uses the theory of regular variation. Suppose f: $\mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^k)$ is Borel measurable, where $\mathrm{GL}(\mathbb{R}^k)$ is the Lie group of invertible linear operators on $\mathbb{R}^k$ with the usual topology. We say that f is *regularly varying* if

$$\lim_{\lambda\to\infty} f(\lambda s) f(\lambda)^{-1} = \psi(s) \in \mathrm{GL}(\mathbb{R}^k) \tag{5}$$

for all $s > 0$. In this case Meerschaert (1988) shows that for some linear operator B called the *index* of f we have $\psi(s) = s^B$ for all $s > 0$. Suppose that $\{X(t)\}_{t\geqslant 0}$ is a proper operator-self-similar process on $\mathbb{R}^k$ and that D is an exponent of $\{X(t)\}$. Maejima (1998) shows that the norming operators $A(\lambda)$ in (2) can always be chosen to vary regularly with index $-D$.

For A,B: $\mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^k)$ we write $A \sim B$ if $A(\lambda)B(\lambda)^{-1} \to I$ as $\lambda \to \infty$. This defines an equivalence relation. The following lemma shows that the norming function A in (2) can be replaced by any $B \sim A$, without affecting the limit process.

Lemma 1. *Assume that $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$ and (2) holds. Then for any $B \sim A$ there exists a shift function b: $\mathbb{R}^+ \to \mathbb{R}^k$ such that*

$$\{B(\lambda)Y(\lambda t) + b(\lambda)\}_{t\geqslant 0} \stackrel{\mathrm{d}}{\Rightarrow} \{X(t)\}_{t\geqslant 0} \quad \textit{as } \lambda \to \infty.$$

Proof. Let $b(\lambda) = B(\lambda)A(\lambda)^{-1}a(\lambda)$. Fix any $n \geqslant 1$ and any $0 \leqslant t_1 < \cdots < t_n$ and define for $\lambda > 0$ a mapping $\Psi(\lambda)$: $(\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ by letting $\Psi(\lambda)(x_1,\ldots,x_n) = (B(\lambda)A(\lambda)^{-1}x_1,\ldots,B(\lambda)A(\lambda)^{-1}x_n)$. Then $\Psi(\lambda)$ is a linear operator and since $B \sim A$ it follows that $\Psi(\lambda) \to Id$ as $\lambda \to \infty$, where Id is the identity on $(\mathbb{R}^k)^n$. Moreover, by assumption $(A(\lambda)Y(\lambda t_i) + a(\lambda))_{i=1,\ldots,n} \Rightarrow (X(t_i))_{i=1,\ldots,n}$ and hence Theorem 2.2.10

of Jurek and Mason (1993) (convergence of operator types) yields

$$(B(\lambda)Y(\lambda t_i)+b(\lambda))_{i=1,\dots,n} = \Psi(\lambda)(A(\lambda)Y(\lambda t_i)+a(\lambda))_{i=1,\dots,n} \Rightarrow Id(X(t_i))_{i=1,\dots,n} = (X(t_i))_{i=1,\dots,n},$$

which concludes the proof. □

Proof of Theorem 1. Apply the result of Maejima (1998) to obtain a norming function $A: \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^k)$ which regularly varies with index $-D$ and such that (2) holds. Then apply Theorem 2.4 of Meerschaert and Scheffler (1998) to obtain an invertible linear operator T on $\mathbb{R}^k$ and a regularly varying function $E: \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^k)$ such that $A(\lambda) \sim E(\lambda)T$ as $\lambda \to \infty$, where every subspace $V_1 \cdots V_p$ in the spectral decomposition of D is $E(\lambda)$-invariant for every $\lambda > 0$. Now by Lemma 1 we get for some shift function $e: \mathbb{R}^+ \to \mathbb{R}^k$ that

$$\{E(\lambda)TY(\lambda t)+e(\lambda)\}_{t\geqslant 0} \stackrel{\mathrm{d}}{\Rightarrow} \{X(t)\}_{t\geqslant 0} \quad \text{as } \lambda \to \infty, \tag{6}$$

proving the first assertion of the theorem. Note $e(\lambda) = E(\lambda)TA(\lambda)^{-1}a(\lambda)$.

Now use (1) along with the fact that $c^{T^{-1}DT} = T^{-1}c^DT$ to see that

$$\begin{aligned}\{T^{-1}X(ct)\} &\stackrel{\mathrm{d}}{=} \{T^{-1}(c^DX(t)+d(c))\}\\ &= \{(T^{-1}c^DT)T^{-1}X(t)+T^{-1}d(c))\}\\ &= \{c^{T^{-1}DT}(T^{-1}X(t))+T^{-1}d(c))\}\end{aligned}$$

for all $c > 0$. It follows that $\{T^{-1}X(t)\}_{t\geqslant 0}$ is a proper operator-self-similar process with exponent $T^{-1}DT$. From (6) along with Theorem 2.2.10 of Jurek and Mason (1993) we obtain

$$\{(T^{-1}E(\lambda)T)Y(\lambda t)+\tilde{e}(\lambda)\}_{t\geqslant 0} \stackrel{\mathrm{d}}{\Rightarrow} \{T^{-1}X(t)\}_{t\geqslant 0} \quad \text{as } \lambda \to \infty,$$

where $\tilde{e}(\lambda) = T^{-1}E(\lambda)TA(\lambda)^{-1}a(\lambda)$. Hence $\{Y(t)\} \in \mathrm{GDOA}(\{T^{-1}X(t)\})$. Define $W_i = T^{-1}(V_i)$ and note that $V_i = T(W_i)$. Then $W_1 \oplus \cdots \oplus W_p$ is the spectral decomposition of $\mathbb{R}^k$ with respect to the exponent $T^{-1}DT$, and $(T^{-1}E(\lambda)T)(W_i) = T^{-1}E(\lambda)(V_i) = T^{-1}(V_i) = W_i$, so that the subspaces W_i are all $T^{-1}E(\lambda)T$-invariant for every $\lambda > 0$. This shows that $\{Y(t)\}$ is spectrally compatible with $\{T^{-1}X(t)\}$, which concludes the proof. □

3. Applications to operator-self-similar processes

In this section we apply the spectral decomposition theorem for operator-self-similar processes to obtain sharp bounds on the growth of these processes over time. We say that $\{X(t)\}$ is strictly operator-self-similar if the centering function $d(c)$ in (1) is identically zero. Hudson and Mason (1982) show that, for any proper operator-self-similar process $\{X(t)\}$, there exists a constant $x \in \mathbb{R}^k$ such that $\{X(t)-x\}$ is strictly operator-self-similar. For the remainder of this section we will assume that $\{X(t)\}$ is strictly operator-self-similar, and this entails no loss of generality.

The fact that $\{X(t)\}$ is continuous in law at $t=0$ restricts the real spectrum of the exponent D. Let $a_1 < \cdots < a_p$ denote the real spectrum of the exponent D. Hudson and Mason (1982) show that every $a_1 \geqslant 0$. If $a_1 = 0$ then, by Theorem 4 of Hudson and Mason (1982), $\{X_1(t)\}$ has constant sample paths. On the other hand, Maejima and Mason (1994) show that if $a_1 > 0$, then $X(0) = 0$ almost surely. Then $X(t) \to 0$ in probability as $t \to 0$. The following simple result implies that $X(t)$ grows faster than $t^{a_1-\varepsilon}$ for t near zero, and slower than $t^{a_p+\varepsilon}$ for t large, for any $\varepsilon > 0$.

Theorem 2. *Suppose that $\{X(t)\}_{t\geqslant 0}$ is a proper strictly operator-self-similar process on $\mathbb{R}^k$ with exponent D, and let $0 < a_1 < \cdots < a_p$ denote the real spectrum of the exponent D:*
(i) *If $\alpha < a_1$ then $Y(t) = t^{-\alpha}X(t)$ is a proper strictly operator-self-similar process with exponent $D - \alpha I$;*
(ii) *If $\beta > a_p$ then the stochastic process defined by $Z(t) = t^{\beta}X(1/t)$ for $t > 0$, and $Z(0)=0$ almost surely, is a proper strictly operator-self-similar process with exponent $\beta I - D$.*

Proof. If $\alpha < a_1$ then from (1) with $d(c)=0$ we obtain

$$\begin{aligned}\{Y(ct)\} &= \{(ct)^{-\alpha}X(ct)\}\\ &\stackrel{\mathrm{d}}{=} \{c^{-\alpha}t^{-\alpha}c^{D}X(t)\}\\ &= \{c^{D-\alpha I}t^{-\alpha}X(t)\}\\ &= \{c^{D-\alpha I}Y(t)\}\end{aligned}$$

for any $c > 0$, so that $\{Y(t)\}$ is strictly operator-self-similar with exponent $D - \alpha I$. If $\beta > a_p$ then

$$\begin{aligned}\{Z(ct)\} &= \{(ct)^{\beta}X(c^{-1}t^{-1})\}\\ &\stackrel{\mathrm{d}}{=} \{c^{\beta}t^{\beta}c^{-D}X(t^{-1})\}\\ &= \{c^{\beta I-D}t^{\beta}X(1/t)\}\\ &= \{c^{\beta I-D}Z(t)\}\end{aligned}$$

for all $c > 0$ so that $\{Z(t)\}$ is strictly operator-self-similar with exponent $\beta I - D$. Note that $c^{\beta I-D} \to 0$ in norm as $c \to 0$ so that $Z(t)$ is continuous in law with $Z(0)=0$. This completes the proof.

Now we will apply the spectral decomposition to sharpen the results of Theorem 2. This result implies that the spectral component $X_i(t)$ grows faster than $t^{a_i-\varepsilon}$ for t near zero, and slower than $t^{a_i+\varepsilon}$ for t large, for any $\varepsilon > 0$. Hence the real spectrum of the exponent D characterizes the growth rate of the process.

Corollary 1. *Suppose that $\{X(t)\}_{t\geqslant 0}$ is a proper strictly operator-self-similar process with exponent D. Let $V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition of $\mathbb{R}^k$ with respect*

to D and $0 < a_1 < \cdots < a_p$ *the real spectrum of D. Write* $X(t) = X_1(t) + \cdots + X_p(t)$ *and* $D = D_1 \oplus \cdots \oplus D_p$ *with respect to this direct sum decomposition. Then*:
(i) *If* $\alpha < a_i$ *then* $Y_i(t) = t^{-\alpha} X_i(t)$ *is a proper strictly operator-self-similar process with exponent* $D_i - \alpha I$;
(ii) *If* $\beta > a_i$ *then the stochastic process defined by* $Z_i(t) = t^{\beta} X_i(1/t)$ for $t > 0$, *and* $Z_i(0) = 0$ *almost surely, is a proper strictly operator-self-similar process with exponent* $\beta I - D_i$.

Proof. Apply Theorem 2 to the proper strictly operator-self-similar component process $\{X_i(t)\}$, which is spectrally simple with exponent D_i where every eigenvalue of D_i has real part equal to a_i.

We can also use the spectral decomposition to obtain sharp upper and lower bounds on the growth rate of the norm and radial projections of the process $\{X(t)\}$. Since these results can also be obtained for stochastic processes in the generalized domain of attraction of a proper operator-self-similar process by essentially the same methods, we will now proceed with the more general case. Every proper operator-self-similar process belongs to its own generalized domain of attraction, and is spectrally compatible with itself. Therefore, sharp upper and lower bounds on $\|X_i(t)\|$ and $|\langle X(t), \theta \rangle|$ can be deduced from the results in the next section by letting $\{Y(t)\} = \{X(t)\}$.

4. Applications to generalized domains of attraction

Now suppose that $\{Y(t)\}_{t \geqslant 0}$ belongs to the generalized domain of attraction of some proper operator-self-similar process. Apply Theorem 1 to obtain a limiting process such that $\{X(t)\}$ and $\{Y(t)\}$ are spectrally compatible. Recall that for any proper operator-self-similar process $\{X(t)\}$, there exists a constant $x \in \mathbb{R}^k$ such that $\{X(t) - x\}$ is strictly operator-self-similar. Replacing the centering constants $a(\lambda)$ in (2) by $a(\lambda) - x$ yields a strictly operator-self-similar limiting process. Letting $t = 0$ in (2) we see that $A(\lambda)Y(0) + a(\lambda) \Rightarrow X(0)$ where the limit is almost surely equal to zero. Since $A(\lambda) \to 0$ in norm we also have $a(\lambda) \to 0$, and so we obtain the same limit with $a(\lambda) = 0$ for all $\lambda > 0$. Then without loss of generality we can take both $d(c)$ in (1) and $a(\lambda)$ in (2) to be identically zero. Now we can obtain sharp bounds on the growth of the process $\{Y(t)\}$ over time. We will obtain bounds on both the norm of the process, and the radial projections. First we must prove a simple lemma concerning the growth rate of regularly varying sequences of linear operators.

Lemma 2. *Suppose that* $f : \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^k)$ *varies regularly with index E.*
(i) *If every eigenvalue of E has negative real part then* $\|f(t)\| \to 0$;
(ii) *If every eigenvalue of E has positive real part then* $\|f(t)x\| \to \infty$ *uniformly on compact subsets of* $\mathbb{R}^k \backslash \{0\}$.

Proof. If every eigenvalue of E has negative real part then $\|\lambda^E\| \to 0$ as $\lambda \to \infty$, see for example Hirsch and Smale (1974, Chapter 6). Choose $\lambda_0 > 1$ such that $\|\lambda_0^E x\| < \frac{1}{2}$ for all $\|x\| = 1$. Meerschaert (1988) shows that the convergence in (5) is uniform on compact subsets of $0<\lambda<\infty$. Given $0<\varepsilon<\frac{1}{2}$ choose $t_0>0$ such that $\|f(\lambda t)f(t)^{-1}-\lambda^E\|<\varepsilon$ for all $t\geqslant t_0$ and all $1\leqslant\lambda\leqslant\lambda_0$. Given $t\geqslant t_0$ write $t=\lambda\lambda_0^n t_0$ where $n\geqslant 0$ is an integer and $1\leqslant\lambda\leqslant\lambda_0$. Since $\|\lambda^E\|$ is a continuous function of $\lambda>0$ there is some $C_1>0$ such that $\|\lambda^E\|\leqslant C_1$ for all $1\leqslant\lambda\leqslant\lambda_0$. Since $\|f(t_0)y\|$ is a continuous function of y, we can choose $M>0$ such that $\|f(t_0)y\|\leqslant M$ for all $\|y\|=1$. Given $\|y\|=1$ let $x=f(t_0)y$. Then

$$\begin{aligned}\|f(t)y\| &= \|f(t)f(t_0)^{-1}x\|\\ &= \|\|f(\lambda\lambda_0^n t_0)f(\lambda_0^n t_0)^{-1}\, f(\lambda_0^n t_0)f(\lambda_0^{n-1}t_0)^{-1}\cdots f(\lambda_0 t_0)f(t_0)^{-1}x\|\\ &\leqslant \|f(\lambda\lambda_0^n t_0)f(\lambda_0^n t_0)^{-1}\|\cdot\|f(\lambda_0^n t_0)f(\lambda_0^{n-1}t_0)^{-1}\|\cdots\|f(\lambda_0 t_0)f(t_0)^{-1}x\|\\ &< (C+\varepsilon)(1/2+\varepsilon)^n M\end{aligned}$$

which tends to zero as $t\to\infty$, which causes $n\to\infty$. Since this upper bound holds for any unit vector y we have $\|f(t)\|\to 0$ which concludes the proof of (i).

Now we note that $(f(t)^*)^{-1}$ is regularly varying with index $-E^*$. If every eigenvalue of E has positive real part then every eigenvalue of $-E^*$ has negative real part, and so $\|(f(t)^*)^{-1}\|=\|(f(t)^{-1})^*\|=\|f(t)^{-1}\|\to 0$ by (i). Now (ii) follows easily from the fact that $\min\{\|f(t)x\|:\|x\|=1\|\}=1/\|f(t)^{-1}\|$.

Theorem 3. *Suppose that $\{X(t)\}_{t\geqslant 0}$ is a proper strictly operator-self-similar process on $\mathbb{R}^k$ with exponent D, and that $\{Y(t)\}\in \mathrm{GDOA}(\{X(t)\})$. Let $0<a_1<\cdots<a_p$ denote the real spectrum of D. Then*:
(i) *For any $\beta>a_p$, $P[\|Y(t)\|>t^\beta]\to 0$ as $t\to\infty$*;
(ii) *If $P[X(1)=0]=0$ then for any $\alpha<a_1$, $P[\|Y(t)\|<t^\alpha]\to 0$ as $t\to\infty$.*

Proof. If $\beta>a_p$ choose $\varepsilon>0$ so that $\beta-\varepsilon>a_p$. It is easy to check that $t^{\beta-\varepsilon}A(t)$ varies regularly with index $(\beta-\varepsilon)I-D$, where every eigenvalue of $(\beta-\varepsilon)I-D$ has positive real part. Then $t^{\beta-\varepsilon}\|A(t)x\|\to\infty$ as $t\to\infty$ uniformly on $\{x:\|x\|=1\}$ by Lemma 2(ii). Choose $t_0>0$ such that $t^{\beta-\varepsilon}\|A(t)y\|>1$ for all $t\geqslant t_0$ and all $\|y\|=1$. It follows that $t^{-\varepsilon}\|A(t)y\|>t^{-\beta}\|y\|$ for all $t\geqslant t_0$ and all $y\neq 0$. Given $\delta>0$ choose $M>0$ such that $P[\|X(1)\|>M]<\delta$ and $P[\|X(1)\|=M]=0$. Then for all $t\geqslant t_0$ for which $t^\varepsilon>M$ we have

$$\begin{aligned}P[\|Y(t)\|>t^\beta] &= P[t^{-\beta}\|Y(t)\|>1]\\ &\leqslant P[t^{-\varepsilon}\|A(t)Y(t)\|>1]\\ &= P[\|A(t)Y(t)\|>t^\varepsilon]\\ &\leqslant P[\|A(t)Y(t)\|>M]\\ &\to P[\|X(1)\|>M]<\delta\end{aligned}$$

as $t\to\infty$. Since $\delta>0$ can be made arbitrarily small, we see that (i) holds.

If $\alpha<a_1$ choose $\varepsilon>0$ so that $\alpha+\varepsilon<a_1$. Since $t^{\alpha+\varepsilon}A(t)$ varies regularly with index $(\alpha+\varepsilon)I-D$, where every eigenvalue of $(\alpha+\varepsilon)I-D$ has negative real part, we have $t^{\alpha+\varepsilon}\|A(t)\|\to 0$ as $t\to\infty$ by Lemma 2(i). Choose $t_0>0$ such that $t^{\alpha+\varepsilon}\|A(t)y\|<1$ for all $t\geqslant t_0$ and all $\|y\|=1$. It follows that $t^\varepsilon\|A(t)y\|<t^{-\alpha}\|y\|$ for all $t\geqslant t_0$ and all $y\neq 0$.

Given $\delta > 0$ choose $m > 0$ such that $P[\|X(1)\| < m] < \delta$ and $P[\|X(1)\| = m] = 0$. Then for all $t \geqslant t_0$ for which $t^{-\varepsilon} < m$ we have

$$\begin{aligned} P[\|Y(t)\| < t^{\alpha}] &= P[t^{-\alpha}\|Y(t)\| < 1] \\ &\leqslant P[t^{\varepsilon}\|A(t)Y(t)\| < 1] \\ &= P[\|A(t)Y(t)\| < t^{-\varepsilon}] \\ &\leqslant P[\|A(t)Y(t)\| < m] \\ &\to P[\|X(1)\| < m] < \delta \end{aligned}$$

as $t \to \infty$. Since $\delta > 0$ can be made arbitrarily small, we see that (ii) holds.

Corollary 2. *Suppose that $\{X(t)\}_{t \geqslant 0}$ is a proper strictly operator-self-similar process on $\mathbb{R}^k$ with exponent D, and that $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$ is spectrally compatible with $\{X(t)\}$. Let $V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition of $\mathbb{R}^k$ with respect to D and $a_1 < \cdots < a_p$ the real spectrum of D. Write $Y(t) = Y_1(t) + \cdots + Y_p(t)$ with respect to this direct sum decomposition. Then*:
(i) *For any $\beta > a_i$, $P[\|Y_i(t)\| > t^{\beta}] \to 0$ as $t \to \infty$;*
(ii) *If $P[X_i(1) = 0] = 0$ then for any $\alpha < a_i$, $P[\|Y_i(t)\| < t^{\alpha}] \to 0$ as $t \to \infty$.*

Proof. Write $D = D_1 \oplus \cdots \oplus D_p$ with respect to the direct sum decomposition $\mathbb{R}^k = V_1 \oplus \cdots \oplus V_p$. Apply Theorem 3 along with the fact that $\{Y_i(t)\} \in \mathrm{GDOA}(\{X_i(t)\})$. Note that $\{X_i(t)\}$ is a proper strictly operator-self-similar process with exponent D_i and that every eigenvalue of D_i has real part equal to a_i.

Remark 1. Since $X(ct)$ has the same distribution as $c^D X(t)$, the condition $P[X(1) = 0] = 0$ implies that $P[X(t) = 0] = 0$ for all $t > 0$. It is possible that $X(1) = 0$ with positive probability. For example, let $\{B(t)\}$ be a standard Gaussian process on $\mathbb{R}^k$ independent of Z Bernoulli, and $X(t) = ZB(t)$. It is also possible that $X(t)$ is confined to a lower-dimensional hyperplane with positive probability for all $t > 0$. For example, take $B_1(t)$ and $B_2(t)$ independent univariate Brownian motions, independent of Z Bernoulli, and let $X(t) = uZB_1(t) + v(1 - Z)B_2(t)$ where $u, v \in \mathbb{R}^2$ are linearly independent.

Remark 2. The exponent of an operator-self-similar process is not unique in general, because of possible symmetries. Theorem 2 of Hudson and Mason (1982) characterizes the set of exponents for a given operator-self-similar process. Using this characterization, it is possible to show that every exponent has the same real spectrum. The proof of this fact can be obtained along the same lines as Theorem 1 and its Corollary in Meerschaert and Veeh (1993), using Lemma 2 of Maejima (1998) to show the existence of a commuting exponent.

Our next two results refine Theorem 3 to yield the growth rate of $Y(t)$ in any radial direction. We begin with two simple lemmas.

Lemma 3. *If $X_n \Rightarrow X$ on $\mathbb{R}^k$ and if $\theta_n \to \theta \neq 0$ and $M_n \to \infty$ then*

$$P[|\langle X_n, \theta_n \rangle| > M_n] \to 0.$$

Proof. It follows from Theorem 2.2.10 of Jurek and Mason (1993) that $|\langle X_n, \theta_n\rangle| \Rightarrow |\langle X, \theta\rangle|$ as $n \to \infty$. Hence the sequence $(|\langle X_n, \theta_n\rangle|)$ is uniformly tight, so given $\varepsilon > 0$ there exists an $R > 0$ such that $P[|\langle X_n, \theta_n\rangle| > R] < \varepsilon$ for all n. Then for all n such that $M_n > R$ we have $P[|\langle X_n, \theta_n\rangle| > M_n] \leqslant P[|\langle X_n, \theta_n\rangle| > R] < \varepsilon$ which concludes the proof.

Lemma 4. *If $X_n \Rightarrow X$ on $\mathbb{R}^k$ and $P[\langle X, \theta\rangle = 0] = 0$ then for any $\theta_n \to \theta \neq 0$ and $\varepsilon_n \to 0$ we have*

$$P[|\langle X_n, \theta_n\rangle| < \varepsilon_n] \to 0.$$

Proof. Given $\delta > 0$ there exists an $m > 0$ such that $P[|\langle X, \theta\rangle| < m] < \delta$ and $P[|\langle X, \theta\rangle| = m] = 0$. Since $\varepsilon_n < m$ for all $n \geqslant n_0$ and some $n_0 \geqslant 1$ we conclude that for $n \geqslant n_0$ by the Portemanteau theorem $P[|\langle X_n, \theta_n\rangle| < \varepsilon_n] \leqslant P[|\langle X_n, \theta_n\rangle| < m] \to P[|\langle X, \theta\rangle| < m] < \delta$ which proves the lemma since $\delta > 0$ can be made arbitrary small.

Theorem 4. *Suppose that $\{X(t)\}_{t \geqslant 0}$ is a proper strictly operator-self-similar process on $\mathbb{R}^k$ with exponent D, and that $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$. Let $0 < a_1 < \cdots < a_p$ denote the real spectrum of D. Then:*
(i) *For any $\theta \neq 0$ and any $\beta > a_p$, $P[|\langle Y(t), \theta\rangle| > t^\beta] \to 0$ as $t \to \infty$;*
(ii) *For any $\theta \neq 0$ and any $\alpha < a_1$, if $X(1)$ has a density then $P[|\langle Y(t), \theta\rangle| < t^\alpha] \to 0$ as $t \to \infty$.*

Proof. Given $t_n \to \infty$ define $t_n^{-\beta}(A(t_n)^*)^{-1}\theta = r_n\omega_n$ where $r_n > 0$ and $\|\omega_n\| = 1$. Then $r_n \to 0$ by Lemma 2(i) since $t^{-\beta}(A(t)^*)^{-1}$ is regularly varying with index $D^* - \beta I$, where every eigenvalue of $D^* - \beta I$ has negative real part. Since ω_n are all unit vectors, for any sequence $t_{n'} \to \infty$ there is a subsequence (n'') along which $\omega_n \to \omega_0$. Then

$$\begin{aligned}
P[|\langle Y(t_n), \theta\rangle| > t_n^\beta] &= P[t_n^{-\beta}|\langle Y(t_n), \theta\rangle| > 1] \\
&= P[|\langle Y(t_n), t_n^{-\beta}\theta\rangle| > 1] \\
&= P[|\langle A(t_n)^{-1}A(t_n)Y(t_n), t_n^{-\beta}\theta\rangle| > 1] \\
&= P[|\langle A(t_n)Y(t_n), (A(t_n)^{-1})^* t_n^{-\beta}\theta\rangle| > 1] \\
&= P[|\langle A(t_n)Y(t_n), r_n\omega_n\rangle| > 1] \\
&= P[|\langle A(t_n)Y(t_n), \omega_n\rangle| > r_n^{-1}] \to 0
\end{aligned}$$

by Lemma 3, since $r_n^{-1} \to \infty$ and $A(t_n)Y(t_n) \Rightarrow X(1)$. This proves (i), since every sequence has a further subsequence with this property.

Now let $t_n^{-\alpha}(A(t_n)^*)^{-1}\theta = r_n\omega_n$ where $r_n > 0$ and $\|\omega_n\| = 1$. Then $r_n \to \infty$ as $t_n \to \infty$ by Lemma 2(ii) since $t^{-\alpha}(A(t)^*)^{-1}$ regularly varies with index $D^* - \alpha I$, where every eigenvalue of $D^* - \alpha I$ has positive real part. For any sequence $t_{n'} \to \infty$ there is a subsequence (n'') along which $\omega_n \to \omega_0$. Then $P[|\langle Y(t_n), \theta\rangle| < t_n^\alpha] = P[|\langle A(t_n)Y(t_n), \omega_n\rangle| < r_n^{-1}] \to 0$ by Lemma 4, since $r_n^{-1} \to 0$ and $A(t_n)Y(t_n) \Rightarrow X(1)$ while $P[\langle X(1), \omega_0\rangle = 0] = 0$. Then (ii) holds, which concludes the proof. The technical condition that $X(1)$ has a density is to ensure that for all $\theta \neq 0$ we have $P[|\langle X(1), \theta\rangle| = 0] = 0$.

Theorem 5. *Suppose that* $\{X(t)\}_{t\geqslant 0}$ *is a proper strictly operator-self-similar process on* $\mathbb{R}^k$ *with exponent* D, *and that* $\{Y(t)\} \in \mathrm{GDOA}(\{X(t)\})$ *is spectrally compatible with* $\{X(t)\}$. *Define* $a^*(\theta)=a_i$ *for* $\theta \in V_1^*\oplus\cdots\oplus V_i^* \setminus V_1^*\oplus\cdots\oplus V_{i-1}^*$, *where* $V_1\oplus\cdots\oplus V_p$ *is the spectral decomposition of* $\mathbb{R}^k$ *with respect to* D *and* $0 < a_1 < \cdots < a_p$ *is the real spectrum of* D. *Then*:

(i) *For any* $\theta \neq 0$ *and any* $\beta > a^*(\theta)$, $P[|\langle Y(t),\theta\rangle| > t^\beta] \to 0$ *as* $t \to \infty$;

(ii) *For any* $\theta \neq 0$ *and any* $\alpha < a^*(\theta)$, *if* $X(1)$ *has a density then* $P[|\langle Y(t),\theta\rangle| < t^\alpha] \to 0$ *as* $t \to \infty$.

Proof. Lemma 2.3 of Meerschaert and Scheffler (1998) shows that $t^{-\beta}\|(A(t)^*)^{-1}\theta\| \to 0$ as $t \to \infty$ for any $\theta \in V_1^* \oplus \cdots \oplus V_i^*$ and that $t^{-\alpha}\|(A(t)^*)^{-1}\theta\| \to \infty$ as $t \to \infty$ for any $\theta \notin V_1^* \oplus \cdots \oplus V_{i-1}^*$. The remainder of the proof is the same as for Theorem 4.

Remark 3. It follows from (1) that if $X(t)$ has a density for some $t > 0$ then it has a density for all $t > 0$. In this case Theorem 5 implies that for any $\varepsilon > 0$ we have

$$P[t^{a^*(\theta)-\varepsilon} < |\langle Y(t),\theta\rangle| < t^{a^*(\theta)+\varepsilon}] \to 1 \tag{7}$$

as $t \to \infty$ for any $\theta \neq 0$. Thus the real spectrum of D governs the growth rate of $Y(t)$ in any radial direction.

5. For further reading

The following references are also of interest to the reader: Bingham et al., 1987; Curtis, 1974.

References

Bingham, N., Goldie, C., Teugels, J., 1987. Regular variation, Encyclopedia of Mathematics and its Applications, Vol. 27. Cambridge University Press, Cambridge.

Curtis, C., 1974. Linear Algebra, 3rd Edition, Allyn and Bacon, Boston.

Hirsch, M., Smale, S., 1974. Differential Equations, Dynamical Systems, and Linear Algebra. Academic Press, New York.

Hudson, W.N., Mason, J.D., 1982. Operator-self-similar processes in a finite-dimensional space. Trans. Amer. Math. Soc. 273, 281–297.

Jurek, Z., Mason, J.D., 1993. Operator-Limit Distributions in Probability Theory. Wiley, New York.

Maejima, M., Mason, J.D., 1994. Operator-self-similar stable processes. Stoch. Proc. Appl. 54, 139–163.

Maejima, M., 1998. Norming operators for operator-self-similar processes. Stochastic Processes and Related Topics. Birkhauser, Boston.

Meerschaert, M., 1988. Regular variation in $\mathbb{R}^k$. Proc. Amer. Math. Soc. 102, 341–348.

Meerschaert, M., 1991. Spectral decomposition for generalized domains of attraction. Ann. Probab. 19, 875–892.

Meerschaert, M., Veeh, J.A., 1993. The Structure of the Exponents and Symmetries of an Operator Stable Measure. J. Theoret. Probab. 6, 713–726.

Meerschaert, M., Scheffler, H.P., 1998. Multivariable regular variation of functions and measures. J. Appl. Anal. 5, 77–98.